# Modern approaches to quantum gravity

## Homework 3

Fall 2025

## 1. The Penrose process

Consider a rotating black hole, described by the Kerr metric,

$$\begin{aligned} ds^2 = &-\left(1-\frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2 \end{aligned} \tag{1}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{2}$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \tag{3}$$

$$a = J/M \tag{4}$$

This black hole has mass $M$ and angular momentum $J$.

(a) Argue that $K^\mu = (\partial/\partial t)^\mu$ and $R^\mu = (\partial/\partial\phi)^\mu$ are Killing vectors of this geometry. Thus, when considering a massive particle following a geodesic motion, characterized by a 4-momentum

$$p^\mu = m\frac{dx^\mu}{d\tau} \tag{5}$$

the quantities $E = -K_\mu p^\mu$ and $L = R_\mu p^\mu$ are conserved.

(b) Asymptotically, $K_\mu$ is time-like, thus the energy $E$ of a particle sitting far away from the black hole is positive. Show that the region where $K_\mu$ goes from time-like to space-like is the ergosphere, characterized by

$$(r - GM)^2 = G^2M^2 - a^2\cos^2\theta \tag{6}$$

Note that this is outside the outer event horizon $r = r_+$, which is characterized by

$$(r_+ - GM)^2 = G^2M^2 - a^2 \tag{7}$$

Thus, inside the ergosphere, it is possible to have **negative energy** with respect to infinity, because $K_\mu$ is spacelike.

(c) This allows for the following process. Consider an object with 4-momentum $p_0^\mu$ sent from infinity, directed into the ergosphere. Assume that, when it is inside the ergosphere, it splits into two objects with 4-momenta $p_1^\mu$ and $p_2^\mu$. By momentum conservation, $p_0^\mu = p_1^\mu + p_2^\mu$. Doting with $K^\mu$ and $R^\mu$ gives energy and angular momentum conservation,

$$E_0 = E_1 + E_2 \qquad L_0 = L_1 + L_2 \tag{8}$$

Then, since $K_\mu$ is space-like in the ergosphere, the energy $-K_\mu \cdot p_2^\mu$ is allowed to be negative. Let us consider a process in which $E_2 < 0$, and the particle $p_2$ falls into the black hole, while $p_1$ escapes to infinity. After this process, the black hole will have its energy and angular momentum modified by $M \to M + \delta M$, $J \to J + \delta J$ with

$$\delta M = E_2 < 0 \qquad \delta J = L_2 \tag{9}$$

This process, called the Penrose process, allows to extract energy from the black hole. However, there is a limitation to this process, because we still need to make sure that $p_2$ moves forward in time when it crosses the event horizon $r_+$. This is done by imposing that

$$p_2^\mu \chi_\mu < 0 \tag{10}$$

where $\chi^\mu = K^\mu + \Omega_H R^\mu$ is the Killing vector, that is null at the horizon. By requiring that $\chi^\mu$ is null at the horizon, show that

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{11}$$

and show that the condition $p_2^\mu \chi_\mu < 0$ implies the limit

$$L_2 < \frac{E_2}{\Omega_H} \tag{12}$$

What does this imply qualitatively on the angular momentum of the final black hole compared to the initial one ?

(d) Determine the induced metric $\gamma_{ij}$ at a fixed time slice, on the horizon $r = r_+$ (remember $\Delta(r_+) = 0$), and compute the area of the outer horizon

$$A = \int \sqrt{|\gamma|} d\theta d\phi \tag{13}$$

You should find

$$A = 4\pi(r_+^2 + a^2)\,. \tag{14}$$

(e) Show that

$$A = 8\pi G^2 \left( M^2 + \sqrt{M^4 - (J/G)^2} \right) \tag{15}$$

and prove that, under infinitesimal variations of $M$ and $J$

$$\delta A = \frac{8\pi G a}{\sqrt{G^2 M^2 - a^2}} (\Omega_H^{-1} \delta M - \delta J) \tag{16}$$

This is precisely the form we expect from the first law. Conclude why the Penrose process of extracting black hole's mass-energy does obey the second law,

$$\delta A > 0\,. \tag{17}$$

(f) What is the maximal amount of energy $E_1 - E_0$ one could possibly extract from a black hole of mass $M$ and spin $J$ ?

## 2. Quantum field theory in Rindler coordinates

The goal of this exercise is to derive the relation between Minkowski and Rindler modes using a method based on analyticity, instead of the Bogoliubov transformations covered in class.

Consider a spacetime described by Rindler coordinates $(r, t)$ with metric

$$ds^2 = -\kappa^2 r^2 dt^2 + dr^2 \,. \tag{18}$$

For simplicity we will restrict our attention to $1+1$ dimensional spacetimes, although generalization to $d+1$ dimensional spacetime is straightforward. This metric can be understood either as describing an observer in Minkowski space experiencing an acceleration $\kappa$, or, equivalently, a stationary observer hovering very close to a black hole horizon whose surface gravity is given by $\kappa$. To study this metric, it is useful to define the Kruskal coordinates $U, V$ and the Minkowskian coordinates $T, X$ by

$$U \equiv -re^{-\kappa t}\,, \qquad V \equiv re^{\kappa t}\,, \tag{19}$$

$$U \equiv T - X\,, \qquad V \equiv T + X\,, \tag{20}$$

such that

$$ds^2 = -dUdV = -dT^2 + dX^2 \,. \tag{21}$$

We will aim at studying quantum field theory from the perspective of the accelerated observer with coordinates $(r, t)$, and then relate it to the Minkowski inertial coordinates $(X, T)$. The Rindler observer at fixed $x$ traces an hyperbolic curve in $(U, V)$ coordinates as depicted below. He thus experiences an effective horizon, the so-called Rindler horizon.

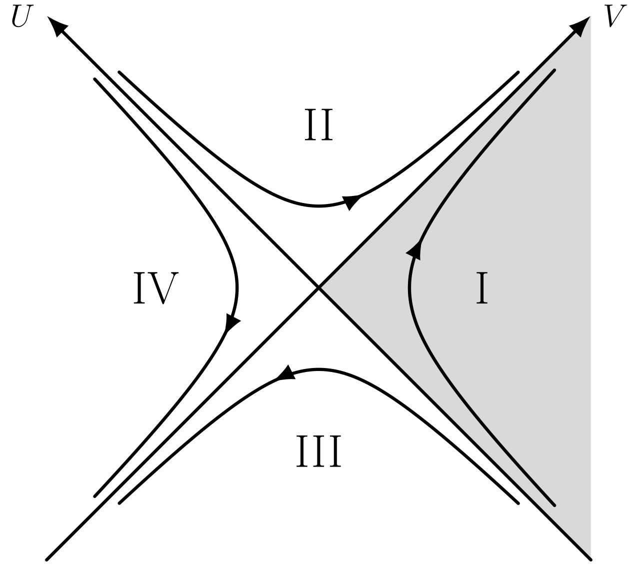

Figure 1: Rindler horizon, Kruskal coordinates, and observers at constant $x$

(a) Consider a massless scalar field. Start by showing that the wave solutions to the Klein-Gordon equation

$$\nabla_a \nabla^a \Phi = 0\,, \tag{22}$$

are given by

$$\Phi \propto e^{-i\omega t} r^{iP} \,, \tag{23}$$

where $P = \pm|\omega|/\kappa$. Restricting to the right wedge $R$ (region I in the diagram), argue that $P > 0$ corresponds to right-moving modes (waves of increasing $r$) whereas $P < 0$ corresponds to left-moving modes (waves of decreasing $r$).

(b) The goal now will be to relate the above modes to the standard positive frequency modes in flat space $e^{-i\omega_0 U} = e^{-i\omega_0 (T-X)}$ and $e^{-i\omega_0 V} = e^{-i\omega_0 (T+X)}$ where $\omega_0 > 0$. To do so, let us first extend our modes from region I to whole space. Converting to Minkowski coordinates, argue that the natural extension is

$$\begin{aligned} u_P^R &\equiv \begin{cases} C_P e^{i\frac{\omega}{\kappa}\log(-U)} & U < 0 \\ 0 & U > 0 \end{cases} \qquad (P > 0)\,, \\ u_P^R &\equiv \begin{cases} C_P e^{-i\frac{\omega}{\kappa}\log V} & V > 0 \\ 0 & V < 0 \end{cases} \qquad (P < 0)\,. \end{aligned} \tag{24}$$

These are the modes with positive frequency in Rindler coordinates, that are non-vanishing in the right Rindler wedge (region I), and vanishing in the left Rindler wedge (region IV). We define the modes $\bar{u}_P^R$ with negative frequency with respect to $t$, by taking the complex conjugate of the above.

Now to complete the picture in all the wedges, we also need the solution that vanishes in the right Rindler wedge and is non-vanishing in the left Rindler wedge. This solution can be easily obtained by using the isometry $(U, V) \to (-U, -V)$, giving

$$\begin{aligned} \bar{u}_P^L &\equiv \begin{cases} C_P e^{i\frac{\omega}{\kappa}\log U} & U > 0 \\ 0 & U < 0 \end{cases} \qquad (P > 0)\,, \\ \bar{u}_P^L &\equiv \begin{cases} C_P e^{-i\frac{\omega}{\kappa}\log(-V)} & V < 0 \\ 0 & V > 0 \end{cases} \qquad (P < 0)\,. \end{aligned} \tag{25}$$

Note that we added a bar compared to (24), this is a choice in convention to account for the fact that in the left Rindler wedge, time runs "backwards" compared to the right Rindler wedge. The modes $u_P^L$ are defined by the complex conjugate of the above.

(c) We will now find what linear combinations of $u_L, \bar{u}_L, u_P, \bar{u}_P$ have positive frequency with respect to $\partial/\partial T$. To do so, consider a right-moving mode $f(U)$. One can expand it in Minkowski modes of frequency $\omega$ by using Fourier analysis,

$$f(U) = \int \frac{d\omega}{2\pi} e^{-i\omega U} \tilde{f}(\omega)\,, \tag{26}$$

where the Fourier inverse is given by

$$\tilde{f}(\omega) = \int dU e^{i\omega U} f(U)\,. \tag{27}$$

For $f$ to have positive frequency with respect to $\partial/\partial T$, we need $\tilde{f}(\omega) = 0$ when $\omega < 0$. Argue that this is satisfied when $f(U)$ is analytic in the lower-half complex plane, and $\lim_{R\to\infty} f(Re^{i\theta}) = 0$, $(\theta \in (-\pi, 0))$.

(d) Let us choose the branch cut of the logarithm to lie on the negative imaginary axis, and let $D_P^{(i)}$ be normalization constants that we will fix later. With this choice of the logarithm, argue that in the case $P > 0$, the combination

$$v_P^{(1)} \equiv D_P^{(1)}(u_P^R + e^{-\frac{\pi\omega}{\kappa}} \bar{u}_P^L) \tag{28}$$

is analytic everywhere in the lower half $U$-plane, and hence represent a mode with positive $\frac{\partial}{\partial T}$ frequency.[1] Argue that this is also true when $P < 0$. Finally, show that

$$v_P^{(2)} \equiv D_P^{(2)}(u_P^L + e^{-\frac{\pi\omega}{\kappa}} \bar{u}_P^R) \tag{29}$$

is a positive frequency mode with respect to $\frac{\partial}{\partial T}$ in both the cases $P > 0$ and $P < 0$.

(e) Noting that the modes $u_P^R$ and $\bar{u}_P^L$ are orthogonal, fix the normalization constants to ensure that the Rindler and Minkowski modes are normalized in the same way, namely $(v_P^{(i)}, v_P^{(i)}) = (u_P^R, u_P^R)$. The Klein-Gordon scalar product $(\alpha, \beta)$ is defined in curved space as

$$(\alpha, \beta) \equiv i \int_{\Sigma_0} d^3x \sqrt{h} n_a (\bar{\alpha} \nabla^a \beta - \beta \nabla^a \bar{\alpha}) , \tag{30}$$

where $\Sigma_0$ is a constant time slice and $n^a$ is a time-forward pointing vector that is normal to $\Sigma_0$.[2] Using this normalization, write $u_P^L$, $\bar{u}_P^L$, $\bar{u}_P^R$, $u_P^R$ in terms of $v_P^{(i)}$. You should find

$$\begin{aligned} u_P^R &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}} (e^{\frac{\pi\omega}{2\kappa}} v_P^{(1)} - e^{-\frac{\pi\omega}{2\kappa}} \bar{v}_P^{(2)}) , \\ u_P^L &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}} (e^{\frac{\pi\omega}{2\kappa}} v_P^{(2)} - e^{-\frac{\pi\omega}{2\kappa}} \bar{v}_P^{(1)}) . \end{aligned} \tag{31}$$

Defining the Rindler annihilation operators as $b_P^R \equiv (u_P^R, \Phi), b_P^L \equiv (u_P^L, \Phi)$ and the standard Minkowski annihilation operator as $a_P^{(i)} = (v_P^{(i)}, \Phi)$, show that

$$\begin{aligned} b_P^R &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}} (e^{\frac{\pi\omega}{2\kappa}} a_P^{(1)} + e^{-\frac{\pi\omega}{2\kappa}} a_P^{(2)\dagger}) , \\ b_P^L &= \frac{1}{\sqrt{2\sinh(\pi\omega/\kappa)}} (e^{\frac{\pi\omega}{2\kappa}} a_P^{(2)} + e^{-\frac{\pi\omega}{2\kappa}} a_P^{(1)\dagger}) . \end{aligned} \tag{32}$$

These are the precious relations between Rindler and Minkowski modes, that allows one to show how the Minkowski vacuum is an entangled state between the right and left Rindler wedges.

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[1] Note that the condition $\lim_{R\to\infty} f(Re^{i\theta}) = 0$ is not satisfied. This is because all the modes we consider (even in Minkowski space) are not normalizable. We will not be concerned with this technical detail.

[2] Note by using the Klein-Gordon equation of motion that the time slice can be continuously deformed without changing the value of the scalar product $(\alpha, \beta)$.

## 3. Radiating Black Holes

(a) Consider a Schwarzschild black hole with the same mass as the sun.
Compute its Hawking temperature and the typical wavelength of the radiation it emits, assuming it radiates like a black body.
Compute its entropy and compare it to the entropy of the sun when it is modelled as a gas of protons and electrons ($10^{35}\,\mathrm{J/K}$).

(b) Using the Stefan-Boltzmann law for blackbody radiation, determine the evaporation time of a black hole with the same mass as the sun.
In light of your previous answers, comment on whether Hawking radiation is detectable with current technology.
**Hint**: you can assume that the black hole only radiates photons.

(c) The heat capacity of a Reissner-Nordström black hole with a mass $M$ and fixed charge $Q$ is defined as:

$$C \equiv T_H \frac{\partial S_{BH}}{\partial T_H}\bigg|_Q \tag{33}$$

Show that it is

$$C = \frac{2S_{BH}\sqrt{M^2-Q^2}}{M-2\sqrt{M^2-Q^2}} \tag{34}$$

(d) Imagine now that the black hole is put into contact with a large heat reservoir at the temperature of the black hole.
By considering small changes of the reservoir's temperature, show that the black holes can be stable if $C > 0$, and they are unstable otherwise.
Show that Reissner-Nordström black holes are stable for masses $|Q| < M < \frac{2|Q|}{\sqrt{3}}$.
Comment on the case of a Schwarzschild black hole.